Incremental Damage Mechanics of Particle or Short-Fiber Reinforced Composites Including Cracking Damage

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In particle or short-fiber reinforced composites, cracking of the reinforcements is a significant damage mode because the cracked reinforcements lose load carrying capacity. This paper deals with an incremental damage theory of particle or short-fiber reinforced composites. The composite undergoing damage process contains intact and broken reinforcements in a matrix. To describe the load carrying capacity of cracked reinforcement, the average stress of cracked ellipsoidal inhomogeneity in an infinite body as proposed in the previous paper is introduced. An incremental constitutive relation on particle or short-fiber reinforced composites including progressive cracking of the reinforcements is developed based on Eshelby's (1957) equivalent inclusion method and Mori and Tanaka's (1973) mean field concept. Influence of the cracking damage on the stress-strain response of composites is demonstrated.

Key Words : Damage Mechanics, Reinforcement Cracking, Particle or Short-Fiber Reinforced Composites, Broken Ellipsoidal Inhomogeneity, Load Carrying Capacity, Equivalent Inclusion Method, Mean Field Theory, Micromechanics

1. Introduction

Particle or short-fiber reinforced composites (hereafter, the composites), have the potential as an engineering material because of their good formability and machinability as well as improved mechanical properties. In these composites, hard particles or short-fibers are dispersed homogeneously in ductile matrices as metal and polymer so as to have more favorable Young's modulus, yield strength, fatigue strength, and resistance to wear, although generally exhibiting poor ductility, low fracture toughness, and strong dependency on processing.

In the composites, a variety of damage modes such as fracture of reinforcements, interfacial debonding between reinforcements and matrix, and cracking in matrix develop from the early stage of deformation under monotonic or cyclic loads (Loretto and Konitzer, 1990 ; Llorca et al., 1993 ; Whitehouse and Clyne, 1993 ; Tohgo et al., 1996). Damage modes are thought to depend on the combination of the mechanical properties of constituents and in-situ interfacial strength. Among them in the composites, the fracture of reinforcements and the interfacial debonding are major damage modes, so that these two modes mainaly affect the mechanical performance of the composites, and seem to be responsible for the low ductility and low fracture toughness. Therefore to extend the application of the composites and to develop even a new composite, thorough understanding of the micromechanics of damage process is essential.

For debonding damage of particle-reinforced composites, Tohgo and Chou (1996) and Tohgo and Weng (1994) developed an incremental damage theory based on the Eshelby's equivalent inclusion method (Eshelby, 1957) and Mori and Tanaka's mean field concept (Mori and Tanaka,

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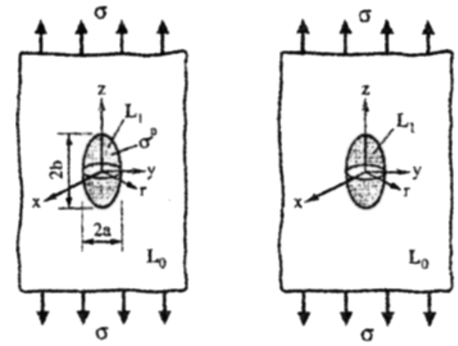
1973), and showed that the influence of the debonding damage on the stress-strain response of the composites is very drastic. On the other hand, the theory for the cracking damage in the composites is also necessary, since the cracking of the reinforcements is commonly observed in composites, particularly such as short-fiber reinforced composites. Many theories (Mura, 1982 ; Arsenault and Taya, 1987 ; Tandon and Weng, 1988 ; Tohgo and Weng, 1994 ; Tohgo and Chou, 1996) for the composites are based on the Eshelby's solution (Eshelby, 1957) for an ellipsoidal inhomogeneity in an infinite body. However, the corresponding solution for a broken ellipsoidal inhomogeneity has not been reported. Therefore, it may be impossible to construct a theory of the composites containing cracking damage in the same scheme.

In the previous paper (Cho et al., 1997a, 1997b ; Cho, 2001), the load carrying capacity of intact and cracked ellipsoidal inhomogeneities in an infinity body was discussed and expressed in terms of the average stress of the intact inhomogeneity and some relevant coefficients. Throght the finite element analyses of the intact and broken ellipsoidal inhomogeneities, the coefficients were given as functions of an aspect ratio for many combinations of the elastic moduli of inhomogeneity and matrix.

In this paper, an incremental constitutive relation of the particle or short-fiber reinforced composites including progressive cracking of the reinforcements is developed by using the Eshelby's equivalent inclusion method and Mori and Tanaka's mean field concept. As a result, influence of the cracking damage on the stress-strain response of the composites is demonstrated.

2. Load Carrying Capacity of a Broken Ellipsoidal Inhomogeneity

Load carrying capacity of an ellipsoidal inhomogeneity embedded in an infinite body can be defined by average stress ; higher average stress than the remote applied stress means high load carrying capacity of the inhomogeneity. Figure 1 (a) shows an intact ellipsoidal inhomogeneity



(a) Intact ellipsoidal inhomogeneity (b) Cracked ellipsoidal inhomogeneity

Fig. 1 Intact and cracked ellipsoidal inhomogeneities in an infinite body subjected to applied stress

embedded in an infinite body under an applied stress $\boldsymbol{\sigma}$. When elastic stiffness tensors of the infinite body (matrix) and the inhomogeneity are denoted by $\boldsymbol{L}_0$ and $\boldsymbol{L}_1$, respectively, the uniform stress of the ellipsoidal inhomogeneity, σ^p, is given by the Eshelby's equivalent inclusion method (Eshelby, 1957) as

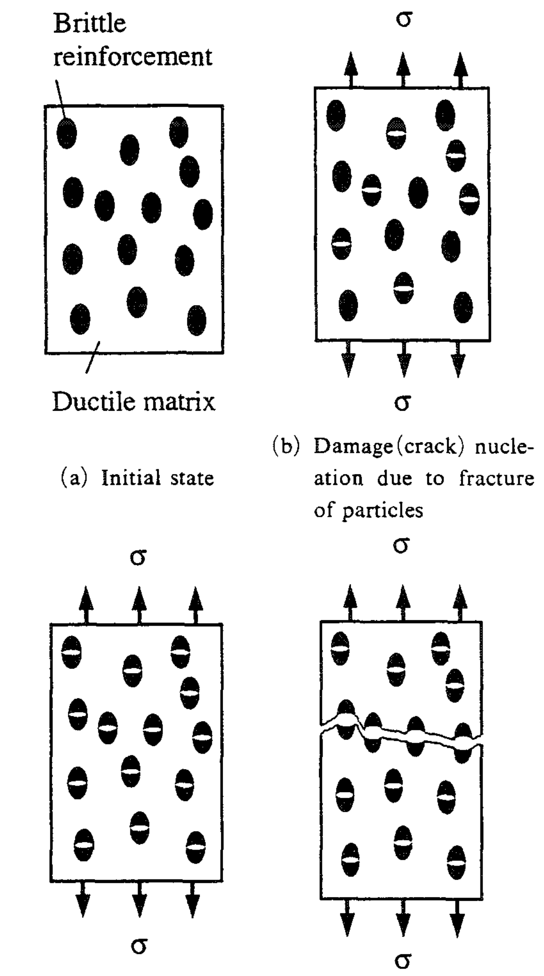
$$\sigma^p = L_0(S-I)[(L_1-L_0)S+L_0]^{-1}L_1(S-I)L_0^{-1}\sigma \quad (1)$$

where $\boldsymbol{S}$ is Eshelby's tensor, a function of shape of inhomogeneity and Poisson's ratio of the matrix, and $\boldsymbol{I}$ is the identity tensor.

For an ellipsoidal inhomogeneity cracked across the cross section on xy-plane as shown in Fig. 1(b), the stress distribution in the inhomogeneity has not been known to the authors' best knowledge. But the average stress in the cracked inhomogeneity, $\bar{\sigma}^{cp}$, may be described as

$$\bar{\boldsymbol{\sigma}}^{cp} = \boldsymbol{\sigma} + \boldsymbol{h}\boldsymbol{\sigma}^p = \boldsymbol{k}\boldsymbol{\sigma}^p \quad (2)$$

where $\boldsymbol{\sigma}^p$ is the stress in intact inhomogeneity given by Eq.(1), $\boldsymbol{h}$ is a coefficient expressing the reduction of average stress due to the cracking damage of an ellipsoidal inhomogeneity and $\boldsymbol{k}$ is the ratio of the average stresses of the cracked inhomogeneity to the intact one. Once the coefficient matrix $\boldsymbol{k}$ is obtained, the average stress of the cracked ellipsoidal inhomogeneity can be easily evaluated. In the previous paper (Cho et al., 1997a, 1997b), the components of $\boldsymbol{k}$ were given as functions for an aspect ratio for a various



(a) Initial state

(b) Damage(crack) nucleation due to fracture of particles

(c) Increase of damage volume fraction due to crack nucleation

(d) Final fracture due to crack coalescence

Fig. 2 Damage process of the particulate-reinforced composite

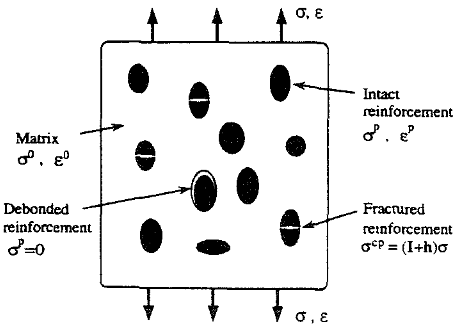


Fig. 3 Microstructure of composite in damage process

of combinations of the elastic moduli of inhomogeneity and matrix.

3. A Damage Process Model

A particle or short-fiber reinforced composite under consideration consists of hard brittle reinforcements and a soft ductile matrix. With high ductile matrix, the damage process of the composite may be described as first incipient crack nucleation due to fracture of particles, then increase of cracking damage volume fraction due to crack nucleation or growth, and finally fracture by crack coalescence, as shown in Fig. 2. However, when the ductility of the matrix is too low, the composite will be fractured by matrix cracking before damage initiation or during damage process. Figure 3 schematically illustrates intact, debonded and cracked particles in the damage process of the composites.

In the composite, the microscopic stresses and strains in the reinforcements and matrix occur due to material heterogeneity, over the applied macroscopic stress and strain. In addition, the cracking of reinforcements leads to the stress release and the reduction of load carrying capacity. In the cracking of a particle, the stress in the particle is not completely released and the particle pantly keeps its load carrying capacity. In this section, an incremental damage theory to describe the progressive cracking damage of reinforcements is developed based on Eshelby's equivalent inclusion method, Mori-Tanaka's mean field concept and the load carrying capacity of broken ellipsoidal inhomogeneity, with following assumptions.

(1) The cracking damage of reinforcements is controlled by the stress on the reinforcements and the statistical behavior of strength of reinforcements.

(2) The stress of cracked reinforcement is released during cracking, and its load carrying capacity is described by the average stress of broken inhomogeneity.

(3) The progressive damage of the composites is described by decrease in intact reinforcements and increase in cracked reinforcements.

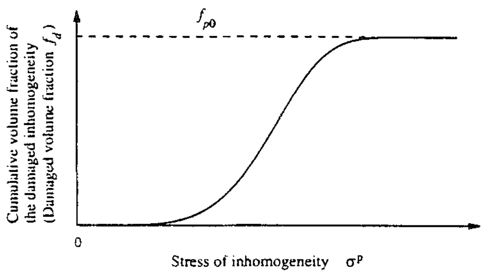


Fig. 4 Cumulative volume fraction of damaged particles as a function of stress of particle

As the composite undergoing the damage process contains the intact and cracked reinforcements, and the reinforcements in the cracking process in the matrix, the cracking of damaged reinforcements is represented by a cumulative distribution of damaged particles as a function of particle stress, as shown in Fig. 4. The damage process of the composite with the initial particle content, f_{p0} under monotonically increasing deformation can be described as follows. First, stresses in the particles and matrix increase with increasing deformation. And when the tensile stress in particles reaches a critical stress of cracking the cracking would start to occur. According to the properties of the cracking stress (Fig. 4), the cracked reinforcements volume fraction increases and the intact reinforcements volume fraction decreases. The composites undergoing the damage process simultaneously contains the intact and cracked reinforcements as well as the reinforcements in the cracking process in the matrix. And the composite behaves like a porous material after all the reinforcements are cracked finally.

The applicability of the present damage model depends on the degree of stress released and the loss of load carrying capacity associated with the damage of reinforcements. And the present model would give a good approximation for the cracking damage of particles in particle and short-fiber reinforced composites because of the loss of load carrying capacity of cracked panticles to a considerable extent.

4. Properties of Reinforcement and Matrix

In this investigation, an incremental constitutive relation of $d\varepsilon_{ij}$ and $d\sigma_{ij}$ is described by a form of being decomposed into the hydrostatic and deviatoric components. The incremental total stress, $d\sigma_{ij}$ and strain, $d\varepsilon_{ij}$ are given by

$$d\sigma_{ij}=d\sigma_{ij}'+\frac{1}{3}d\sigma_{kk}\delta_{ij},\ d\varepsilon_{ij}=d\varepsilon_{ij}'+\frac{1}{3}d\varepsilon_{kk}\delta_{ij} \quad (3)$$

where δ_{ij} is the Kronecker delta and primed componunts mean deviatoric ones. In the composites of this study, spherical particle reinforcement is elastic and matrix is elastic-plastic. The elastic incremental stress-strain relation of the particle is given in isotropic form by

$$d\varepsilon_{kk}=\frac{1}{3\chi_1}d\sigma_{kk},\ d\varepsilon_{ij}'=\frac{1}{2\mu_1}d\sigma_{ij}' \quad (4)$$

where χ_1 and μ_1 are the bulk modulus and the shear modulus of particles. The elastic behavior is also given in the same form with the moduli χ_0 and μ_0 as

$$d\varepsilon_{kk}=\frac{1}{3\chi_0}d\sigma_{kk},\ d\varepsilon_{ij}'=\frac{1}{2\mu_0}d\sigma_{ij}', \quad (5)$$

χ_1, μ_1, χ_0 and μ_0 are related to Young's moduli E_1 and E_0 and Poisson's ratio ν_1 and ν_0 as follows.

$$\chi_i=\frac{E_i}{3(1-2\nu_i)},\ \mu_i=\frac{E_i}{2(1-\nu_i)},\ i=0 \text{ or } 1 \quad (6)$$

The elastic-plastic deformation of matrix is described by the Prandtl-Reuss equation (the J_2-flow theory):

$$d\varepsilon_{kk}=\frac{1}{3\chi_0}d\sigma_{kk},\ d\varepsilon_{ij}'=\frac{1}{2\mu_0}d\sigma_{ij}'+\frac{9\sigma_{ij}'\sigma_{kl}'}{4\sigma_e^2H'}d\sigma_{kl}' \quad (7)$$

where

$$\sigma_e=\left(\frac{3}{2}\sigma_{ij}'\sigma_{ij}'\right)^{1/2},\ d\varepsilon_e^p=\left(\frac{2}{3}d\varepsilon_{ij}^{pl}\ d\varepsilon_{ij}^{pl}\right)^{1/2} \quad (8)$$

and

$$H'=\frac{d\sigma_e}{d\varepsilon_e^p} \quad (9)$$

Hear σ_e and $d\varepsilon_e^p$ are von Mises equivalent stress and incremental equivalent plastic strain, respectively, and $d\varepsilon_{ij}^{pl}$ is the incremental plastic strain. Also, H' is the work-hardening modulus of the matrix. From a comparison of Eqs. (5) and (7), the equivalent shear modulus μ_0' and Poisson's ratio ν_0' in elastic-plastic deformation are approximated as below (Berveiller and Zaoui, 1979), taking Eq.(8) into consideration :

$$\mu_0'=\frac{\mu_0}{1+\frac{3\mu_0}{H'}},\ \nu_0'=\frac{\nu_0+\frac{\mu_0}{H'}(1+\nu_0)}{1+2\frac{\mu_0}{H'}(1+\nu_0)}. \qquad (10)$$

Therefore, Eq.(7) is approximately described by the following isotropic relation :

$$d\varepsilon_{kk}=\frac{1}{3\chi_0}d\sigma_{kk},\ d\varepsilon_{ij}'=\frac{1}{2\mu_0'}d\sigma_{ij}'. \qquad (11)$$

As the above relation is strictly valid in the case of monotonic proportional loading, the relation is used for matrix plasticity in the present theory. In addition the reinforcement is elastic so that the elastic incremental stress-strain relation of the reinforcement is given in the isotropic form :

$$d\boldsymbol{\sigma}=\boldsymbol{L}_1(E_1,\ \nu_1)\,d\boldsymbol{\varepsilon} \qquad (12)$$

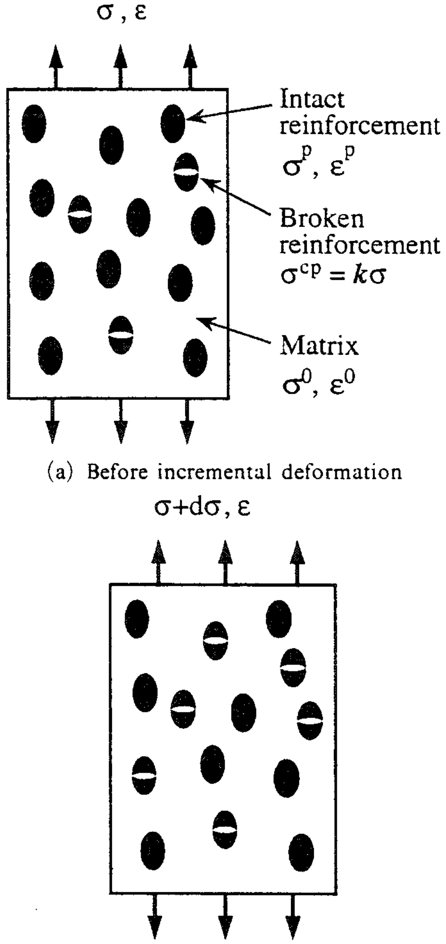
where $L_1(E_1,\ \nu_1)$ is the elastic stiffness described by the Young's modulus, E_1 and Poisson's ratio, ν_1 of the reinforcement. The elastic behavior of the matrix is also taken to be isotropic with the elastic stiffness, $\boldsymbol{L}_0(E_0,\ \nu_0)$. The Prandtl-Reuss equation to describe the elastic-plastic deformation of the matrix is approximated by the same form as the elastic relation :

$$d\boldsymbol{\sigma}=\boldsymbol{L}_0(E_0',\ \nu_0')\,d\boldsymbol{\varepsilon} \qquad (13)$$

where $\boldsymbol{L}_0(E_0',\ \nu_0')$ is the equivalent tangent modulus obtained by replacing E_0 and ν_0 with E_0' and ν_0' for $\boldsymbol{L}_0(E_0,\ \nu_0)$. Where E_0' and ν_0' are the equivalent Young's modulus and Poisson's ratio under the elastic-plastic deformation and given by

$$E_0'=\frac{E_0}{1+\frac{E_0}{H'}},\ \nu_0'=\frac{\nu_0+\frac{E_0}{2H'}}{1+\frac{E_0}{H'}}. \qquad (14)$$

Equation (13) is strictly valied in the case of monotonic proportional loading. In the study the stress and strain of the intact and damaged reinforcements and matrix are represented as the superscripts p, d and 0, respectively, and those of the composite are used by symbols without superscript.



(a) Before incremental deformation

(b) After incremental deformation

Fig. 5 The states of composite undergoing damage process before and after incremental deformation. df_p is a volume fraction of the reinforcements damaged in the incremental deformation

5. Incremental Damage Mechanics of Progressive Cracking Damage

5.1 Formulation based on Eshelby's equivalent inclusion method

Figure 5 illustrates the states before and after incremental deformation of the composite in damage process. The before deformation state in

Fig. 5(a) is described in terms of the volume fractions of the intact and damaged reinforcements, f_p and f_d. If the volume fraction of the reinforcements cracked during incremental deformation is denoted by df_p, the after deformation state in Fig. 5(b) is described in terms of the volume fractions of the intact and damaged reinforcements $f_p - df_p$ and $f_p + df_p$. In order to describe the deformation and damage of the composite in this incremental process, Eshelby's equivalent inclusion method and Mori and Tanaka's mean field theory are used for heterogeneous body containing the intact and damaged reinforcements and the reinforcements to be cracked.

Following Eshelby's equivalence principle combined with Mori and Tanaka's mean field concept (Mura, 1982 ; Arsenault and Taya, 1987 ; Tandon and Weng, 1988), the incremental stress in the reinforcements $d\sigma^p$ is given by

$$\begin{aligned} d\sigma^p &= d\sigma + d\tilde{\sigma} + d\sigma_1^{pt} = L_1(d\varepsilon_0 + d\tilde{\varepsilon} + d\varepsilon_1^{pt}) \\ &= L_0(d\varepsilon_0 + d\tilde{\varepsilon} + d\varepsilon_1^{pt} - d\varepsilon_1^*). \end{aligned} \tag{15}$$

Since the average stress of the cracked reinforcements is given by Eq. (2), Eshelby's equivalence principle for the cracked reinforcements can be written as

$$\begin{aligned} kd\sigma^p &= d\sigma + d\tilde{\sigma} + d\sigma_2^{pt} \\ &= kL_1(d\varepsilon_0 + d\tilde{\varepsilon} + d\varepsilon_1^{pt}) \\ &= L_0(d\varepsilon_0 + d\tilde{\varepsilon} + d\varepsilon_2^{pt} - d\varepsilon_2^*). \end{aligned} \tag{16}$$

Furthermore, for the reinforcements in cracking process, since the current reinforcement stress should be released up to $k\sigma^p$ in the next incremental deformation, the next equation is obtained:

$$\begin{aligned} -(I-k)\,\sigma^p &= d\sigma + d\tilde{\sigma} + \sigma_3^{pt} \\ &= L_0(d\varepsilon_0 + d\tilde{\varepsilon} + d\varepsilon_3^{pt} - \varepsilon_3^*). \end{aligned} \tag{17}$$

Neglecting the high-order terms of increments, the above relation becomes

$$-(I-k)\,\sigma^p = \sigma_3^{pt} = L_0(\varepsilon_3^{pt} - \varepsilon_3^*) \tag{18}$$

In the above equations, L_1 and L_0 are the tangential modulus tensors for the reinforcements and matrix, respectively. $d\sigma$ and $d\tilde{\sigma}$ are the incremental applied stress and the incremental average stress based on Mori and Tanaka's mean field concept, and they are related to $d\varepsilon_0$ and $d\tilde{\varepsilon}$ by

$$d\sigma = L_0 d\varepsilon_0,\ d\tilde{\varepsilon} = L_0 d\tilde{\varepsilon}. \tag{19}$$

$d\sigma_1^{pt}$, $d\sigma_2^{pt}$, σ_3^{pt} and $d\varepsilon_1^{pt}$, $d\varepsilon_2^{pt}$, ε_3^{pt} represent the perturbed parts of the stress and strain in the intact and damaged reinforcements and the reinforcements to be cracked, respectively. $d\varepsilon_1^*$, $d\varepsilon_2^*$ and ε_3^* are Eshelby's equivalent transformation strains. The perturbed strains are related to the transformation strains:

$$d\varepsilon_1^{pt} = Sd\varepsilon_1^*,\ d\varepsilon_2^{pt} = Sd\varepsilon_2^*,\ \varepsilon_3^{pt} = S\varepsilon_3^*. \tag{20}$$

S is Eshelby's tensor for a ellipsoidal inclusion. $d\sigma_1^{pt}$, $d\sigma_2^{pt}$ and σ_3^{pt} are described by

$$\begin{aligned} d\sigma_1^{pt} &= L_0(S-I)\,d\varepsilon_1^*, \\ d\sigma_2^{pt} &= L_0(S-I)\,d\varepsilon_2^*, \\ \sigma_3^{pt} &= L_0(S-I)\,\varepsilon_3^*, \end{aligned} \tag{21}$$

where I is the fourth-rank identity tensor. Since the incremental applied (macroscopic) stress $d\sigma$ is represented by

$$\begin{aligned} d\sigma &= (f_d - df_d)\,d\sigma^p + f_d k d\sigma^p - df_p(I-k)\,\sigma^p \\ &\quad + (1-f_p-f_d)\,(d\sigma + d\tilde{\sigma}), \end{aligned} \tag{22}$$

neglecting the high-order terms of increments, the incremental average stress $d\tilde{\sigma}$ is given by

$$d\tilde{\sigma} = -f_p d\sigma_1^{pt} - f_d d\sigma_2^{pt} - df_p \sigma_3^{pt} \tag{23}$$

Substituting Eqs. (19) and (23) into Eq.(22), one obtains Eq.(24) for the incremental average strain.

$$d\tilde{\varepsilon} = -(S-I)\,(f_p\ d\varepsilon_1^* - f_d\ d\varepsilon_2^* - df_p\ \varepsilon_3^*) \tag{24}$$

The incremental macroscopic strain of the composite is expressed as

$$\begin{aligned} d\varepsilon &= (f_d - df_d)\,(d\varepsilon_0 + d\tilde{\varepsilon} + d\varepsilon_1^{pt}) \\ &\quad + f_d(d\varepsilon_0 + d\tilde{\varepsilon} + d\varepsilon_2^{pt}) \\ &\quad + df_p(d\varepsilon_0 + d\tilde{\varepsilon} + \varepsilon_3^{pt}) + (1-f_p-f_d)\,(d\varepsilon_0 + d\tilde{\varepsilon}). \end{aligned} \tag{25}$$

Considering Eqs. (20) and (24) and neglecting high-order terms of increments, the above equation becomes

$$d\varepsilon = d\varepsilon_0 + f_p\ d\varepsilon_1^* + f_d\ d\varepsilon_2^* + df_p\ \varepsilon_3^* \tag{26}$$

By solving the above equations, Eshelby's equivalent transformation strains $d\varepsilon_1^*$, $d\varepsilon_2^*$ and ε_3^* can be described as a function of incremental applied stress $d\sigma$ and df_p. Finally, the incremental strain $d\varepsilon$-stress $d\sigma$ relation of the composite is obtained as

$$d\varepsilon=(I-f_d\,A_1^{-1}B_1+f_d\,A_2^{-1}B_2)L_0^{-1}d\sigma+[(I-S)^{-1}+f_pA_1^{-1}B_1+f_dA_2^{-1}B_2]L_0^{-1}(I-k)\sigma^p df_p \quad (27)$$

where

$$A_1=(L_1-L_0)^{-1}[L_0+(L_1-L_0)S+(L_1-L_0)(I-S)f_d]-f_d[L_0+(kL_1-L_0)f_d]^{-1}[kL_1S+(kL_1-L_0)(I-S)f_d], \quad (28)$$

$$B_1=f_d[L_0+(kL_1-L_0)f_d]^{-1}+(kL_1-L_0)-I, \quad (29)$$

$$A_2=[L_0+(L_1-L_0)S+(L_1-L_0)(I-S)f_d]^{-1}(L_1-L_0)(I-S)f_d-[kL_1S+(kL_1-L_0)(I-S)f_p]^{-1}[L_0+(kL_1-L_0)f_d](I-S), \quad (30)$$

$$B_2=[kL_1S+(kL_1-L_0)(I-S)f_p]^{-1}+(kL_1-L_0)-[L_0+(L_1-L_0)S+(L_1-L_0)(I-S)f_p]^{-1}(L_1-L_0). \quad (31)$$

As shown in Eq. (27), the composite strain increment is given by the sum of the strain increment due to the stress increment and the strain increment due to the cracking damage. The incremental average stresses of the matrix and the intact and damaged reinforcements, $d\sigma^0$, $d\sigma^p$ and $d\sigma^d$, are given by

$$d\sigma^0=L_0(I-S)[(I-S)^{-1}+f_pA_1^{-1}B_1+f_dA_2^{-1}B_2)L_0^{-1}\times[d\sigma+(I-k)\sigma^p df_p], \quad (32)$$

$$d\sigma^p=L_0(I-S)[(I-S)^{-1}-(1-f_p)A_1^{-1}B_1+f_dA_2^{-1}B_2]L_0^{-1}\times[d\sigma+(I-k)\sigma^p df_p], \quad (33)$$

and

$$d\sigma^d=kd\sigma^p. \quad (34)$$

5.2 Volume fraction of Reinforcements in cracking process

The following Weibull distribution is adopted for the cumulative probability of fracture of reinforcements :

$$P_v\sigma^p{}_{\max}=1-\exp\left[-\left(\frac{\sigma^p{}_{\max}}{S_0}\right)^m\right] \quad (35)$$

where $\sigma^p{}_{\max}$ is the maximum tensile stress of the reinforcements and S_0 and m are the scale and shape parameters, respectively. The average strength of the reinforcements is given by

$$\bar{\sigma}^p{}_{\max}=S_0\Gamma\left(1+\frac{1}{m}\right) \quad (36)$$

where $\Gamma(\cdot)$ is Gamma function. With an initial reinforcement volume fraction, f_{p0}, as a cumulative volume fraction of the damaged reinforcements is represented by $f_{p0}P_v$, the volume fraction df_p of the reinforcements to be cracked in the incremental deformation is given by

$$df_p=f_{p0}\frac{dP_v}{d\sigma^p{}_{\max}}d\sigma^p{}_{\max}. \quad (37)$$

5.3 Equivalent stress of matrix in composite

In order to describe the matrix plasticity, we need the equivalent stress of the matrix in composite. However, it is difficult to estimate the equivalent stress of the heterogeneously deformed matrix in the composite. Tohgo and Weng (1994) extended the Qui and Weng's energy approach (1992), and proposed the expression of the equivalent stress of the matrix for the incremental theory of the composite with debonding damage. It is easy to apply this approach to the composite with reinforcement cracking damage.

An initial equivalent stress, $\sigma_e{}^0$ of the matrix in the composite before plastic deformation and damage is given by

$$(\sigma_e{}^0)^2=\frac{3E_0}{2(1-f_{po})(1+\nu_0)}(2U-f_{po}\sigma^p\varepsilon^p)-\frac{9(1-2\nu_0)}{2(1+\nu_0)}(\sigma_m{}^0)^2 \quad (38)$$

where $\sigma_m{}^0$ is the average hydrostatic stress of matrix, and σ^p and ε^p are the stress and strain of reinforcement. U is the energy of a unit volume of the composite

$$U=\frac{1}{2}\sigma\varepsilon \quad (39)$$

After the incremental deformation the equivalent stress of the matrix is estimated by $\sigma_e{}^0+d\sigma_e{}^0$, where $\sigma_e{}^0$ and $d\sigma_e{}^0$ denote the current equivalent stress before the incremental deformation and its increment, respectively. In the numerical analysis $\sigma_e{}^0$ is known and $d\sigma_e{}^0$ is given by

$$d\sigma_e{}^0=\frac{3E'_0}{2(1-f_p-f_d)(1+\nu'_0)\sigma_e{}^0}\{dU-dR-f_p\sigma^p d\varepsilon^p-f_d\sigma^d d\varepsilon^d+\frac{1}{2}df_p(\sigma^p\varepsilon^p-\sigma^d\varepsilon^d)\}-\frac{9(1-2\nu'_0)}{2(1+\nu'_0)}\sigma_m{}^0 d\sigma_m{}^0 \quad (40)$$

where dU is the incremental energy of the composite and dR is the energy relased by the cracking damage

$$dU = \sigma d\varepsilon, \tag{41}$$

$$dU = \frac{1}{2}\sigma d\varepsilon^d. \tag{42}$$

In Eq. (40), E'_0 and ν'_0 are reduced to their elastic counterparts E_0 and ν_0, when the matrix is in the elastic state.

6. Numerical Results and Discussion

As an example of application of the present incremental damage theory, the stress-strain response under uniaxial tension has been calculated on particle or short fiber reinforced composites taking account of progressive cracking damage of the reinforcements. Note that the constituent materials are isotropic elastic. The numerical analyses, based on the present theory in the section 5, are carried out on the stress-strain response of the composites with progressive reinforcement damage. The numerical calculation programs for analysis of elastic-plastic behavior and damage behavior of particle or short-fiber reinforced composites are developed by the author are used. Here the numerical results are separately performed for elastic case and elastic-plastic one that are based on the incremental damage theory. Young's modulus and Poisson's ratio are $E_1 = 2500\sigma_0$, $\nu_1 = 0.17$ for the reinforcements, and $E_0 = 500\sigma_0$, $\nu_0 = 0.30$ for the matrix. The equivalent stress - equivalent plastic strain relation of the matrix is given by

$$\sigma_e{}^0 = \sigma_0\left(1 + \frac{\varepsilon_e^{0p}}{\varepsilon_0}\right)^{0.1}, \quad \varepsilon_0 = \frac{\sigma_0}{E_0} \tag{43}$$

where σ_0 is the yield stress of the matrix. For the cracking damage of the reinforcements, Weibull distribution Eq. (35) with $m=5.0$ and $\bar{\sigma}^p{}_{\max} = 3.0\sigma_0$ is adopted. Initial volume fraction of the reinforcements contained in the composites is 20%.

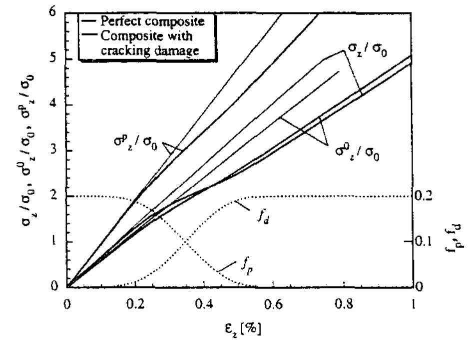


Fig. 6 Stress-strain relation of perfect composite and composite with progressive cracking damage

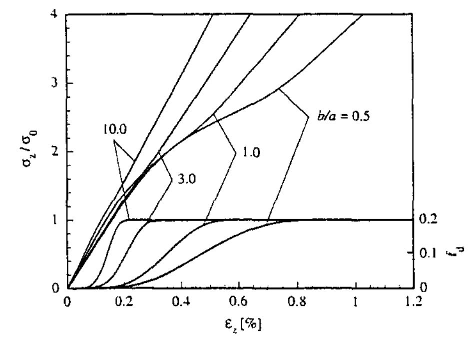


Fig. 7 Influence of an aspect ratio of reinforcements on the stress-strain relations of the elastic composite with progressive cracking damage

6.1 Elastic analysis based on the incremental damage theory

Figures 6, 7 and 8 show the influence of reinforcement damage on the stress-strain behavior of the elastic composite in which both the reinforcement and matrix are elastic. The variations of macroscopic stress (composite stress, σ_z), microscopic stresses of the reinforcements and matrix ($\sigma^p{}_z$ and $\sigma^0{}_z$), volume fractions of intact and damaged reinforcements (f_p and f_d) are shown as a function of the macroscopic composite strain in Fig. 6. Furthermore, the variations of the stresses for the composite without cracking damage are shown by dashed lines.

On the perfect composite, linear stress-strain relations are obtained, and the microscopic stresses are higher in the reinforcements and lower in the matrix than the macroscopic stress. On the composite with progressive damage, the stress-

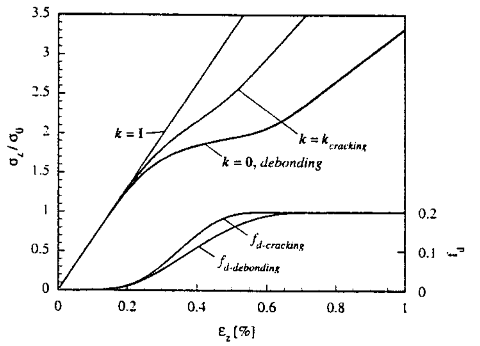


Fig. 8 Influence of $\boldsymbol{k}$-matrix on the stress-strain relation of the elastic composites

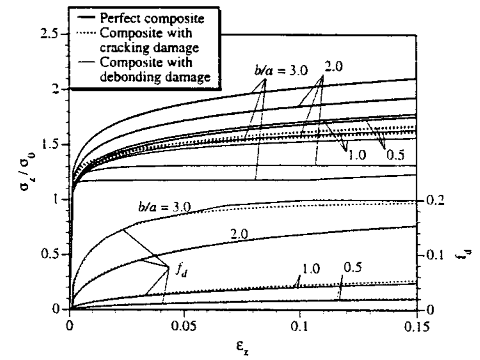


Fig. 9 Influence of an aspect ratio of reinforcements on the elastic-plastic stress-strain relations of the perfect composite and the composites with progressive cracking or debonding damage

strain relations deviate from linear relations for the perfect composite with increasing cracking damage of the reinforcements. After all reinforcements encounter the cracking damage, the stress-strain relations would be linear again. Figure 7 shows the influence of the aspect ratio of the reinforcements on the stress-strain relations of the composites with progressive damage. When the aspect ratio of the reinforcements is lower, the influence of the cracking damage of the reinforcements is more drastic because the reinforcements lose the larger amount of load carrying capacity. The present theory can describe a variety of damage patterns of the reinforcements such as the cracking damage and the debonding damage by modifying the $\boldsymbol{k}$-matrix. Figure 8 shows the influence of the $\boldsymbol{k}$-matrix on the stress-strain relations of the particle reinforced composites. When $\boldsymbol{k}$ is unit matrix, the present theory describes the perfect composite because of no loss of load carrying capacity. On the other hand, since the condition of null matrix for $\boldsymbol{k}$ means the complete loss of load carrying capacity of the reinforcement, the present theory describes the interfacial debonding between the reinforcement and matrix, and corresponds to the previous incremental damage theory for particle reinforced composite with debonding damage (Tohgo and Weng, 1994 ; Tohgo and Chou, 1996). It is noted from Fig. 8 that the debonding damage corresponding $\boldsymbol{k}=0$ gives the lower limit of the stress-strain relation with the reinforcement damage.

6.2 Elastic-plastic analysis based on the incremental damage theory

Figure 9 shows the elastic-plastic stress-strain relation of a perfect composite and a composite with cracking or debonding damage. The stress-strain relation of the composite is affected by the progressive reinforcement damage, and the composite with debonding damage exhibits the lowest stress-strain relation. As the reinforcement with high aspect ratio carries the high stress, it has the reinforcing effect in the perfect composite and enhances the damage evolution in the composite with the reinforcement damage. Furthermore, the reinforcement with high aspect ratio still maintains high load carrying capacity after the cracking damage and completely lose it by the debonding damage. As a result, the influence of the aspect ratio on the stress-strain relation is outstanding in the perfect composite and in the composite with the debonding damage, and is not remarkable in the composite with the cracking damage.

7. Conclusion

An incremental damage theory of particle and short-fiber reinforced composites with progressive reinforcement cracking is developed based on the Eshelby's equivalent method, the Mori-Tanaka's mean field concept, and the load carrying capacity of broken ellipsoidal inhomogeneity. Furthermore, in order to estimate the equivalent stress of the heterogeneously deformed matrix in the composite, the expression fit to the incremental damage theory is also derived based on the Weng-Qiu's energy approach. As a result the present theory can describe not only cracking damage but also debonding damage of reinforcements by modifying the load carrying capacity of damaged reinforcements. And numerical results for the stress-strain relation of the composite under uniaxial tension exhibit that the influence of reinforcement damage on the stress-strain relation of a composite is drastic and depends on the aspect ratio of the reinforcements, and that the full debonding damage gives the lower limit of the stress-strain relation of the composite with the reinforcement damage.

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